Persistence of randomly coupled fluctuating interfaces

Satya N. Majumdar[1,2] and Dibyendu Das[3]

[1]*Laboratoire de Physique Théorique (UMR C5152 du CNRS), Université Paul Sabatier, 31062 Toulouse Cedex, France*
[2]*Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris-Sud, Bât. 100, 91405 Orsay Cedex, France*
[3]*Department of Physics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India*

We study the persistence properties in a simple model of two coupled interfaces characterized by heights h_1 and h_2, respectively, each growing over a d-dimensional substrate. The first interface evolves independently of the second and can correspond to any generic growing interface, e.g., of the Edwards-Wilkinson or of the Kardar-Parisi-Zhang variety. The evolution of h_2, however, is coupled to h_1 via a quenched random velocity field. In the limit $d \rightarrow 0$, our model reduces to the Matheron–de Marsily model in two dimensions. For $d=1$, our model describes a Rouse polymer chain in two dimensions advected by a transverse velocity field. We show analytically that after a long waiting time $t_0 \rightarrow \infty$, the stochastic process h_2, at a fixed point in space but as a function of time, becomes a fractional Brownian motion with a Hurst exponent, $H_2 = 1 - \beta_1/2$, where β_1 is the growth exponent characterizing the first interface. The associated persistence exponent is shown to be $\theta_s^2 = 1 - H_2 = \beta_1/2$. These analytical results are verified by numerical simulations.

I. INTRODUCTION

The survival probability $P(t)$ that a stochastic process $X(t)$ does not cross zero up to time t is a quantity of long standing interest in probability theory and with many practical applications [1]. The derivative $F(t) = -dP/dt$ is called the first-passage probability [2]. This subject has seen a resurgent interest over the last decade in the context of many body nonequilibrium systems where the stochastic process $X(t)$ denotes a local time dependent field in a spatially extended evolving system. For example, in the case of the Ising model evolving under the Glauber dynamics, the relevant stochastic process $X(t)$ is the spin $s_i(t)$ at a fixed site i as a function of time t and $P(t)$ then denotes the probability that the spin s_i does not flip up to time t [3]. In this context, the survival probability $P(t)$ thus measures the "persistence" of a local field to remain in its initial state. In many of these nonequilibrium systems, the persistence has been found to decay as a power law at late times, $P(t) \sim t^{-\theta}$. The exponent θ is called the persistence exponent and has been a subject of much theoretical, numerical, and experimental studies in recent times [4]. The exponent θ is very hard to calculate analytically even in simple systems such as the linear diffusion equation starting from random initial conditions [5]. The reason for this difficulty can be traced back to the fact that the spatial interactions in these extended systems makes the local stochastic field $X(t)$ a "non-Markovian" process in time [4].

Apart from these pure systems, persistence has also been studied in systems with quenched disorder [6,7]. In the presence of disorder, the exact calculation of persistence $P(t)$ is nontrivial even for a single particle without any spatial interaction, though some analytical results have been obtained recently. For example, the asymptotic results for the persistence of a particle moving in a random Sinai potential in one dimension have been obtained both in the case of a vanishing external field [8–10] and also for nonzero external field [11].

Another solvable example is the persistence of a single particle advected by a layered random velocity field [12–14].

The purpose of this paper is to present analytical results for the persistence $P(t)$ in a system with both spatial interaction and disorder. Our system consists of two growing ($d+1$)-dimensional interfaces where d refers to the dimension of the substrate on which the surfaces grow and 1 refers to the time t. The two interfaces are characterized by their heights $h_1(\mathbf{r}, t)$ and $h_2(\mathbf{r}, t)$, respectively. In our model, while the height h_1 of the first interface evolves independently of h_2, the evolution of h_2 is driven by a velocity that is a random quenched function of h_1. This random velocity represents the disorder in the system. The model is detailed in Sec. II.

There are two motivations for studying this model:

(i) We will show later in Sec. III that our model corresponds to different well known physical systems for different values of the spatial dimension d. For example, in the limit $d \rightarrow 0$, our model reduces to the Matheron–de Marsily model where one studies the motion of a single Brownian particle in a two-dimensional plane in the presence of a transverse random velocity field [15]. On the other hand, for $d=1$, we will show that our model describes the evolution of a Rouse polymer chain [16] (a Rouse chain consists of a set of beads or monomers connected by harmonic springs) moving in a two-dimensional plane and advected by a transverse random velocity field. While the transport properties in this latter model are well understood [17–19], the persistence properties are less understood. The analytical results obtained in this paper for the persistence in our model for general d will apply to these models in the limiting cases $d=0$ and $d=1$.

(ii) The persistence properties of a single interface have been studied extensively both theoretically [20–27] and more recently they have been measured experimentally in a system of fluctuating steps on Si-Al surfaces [28]. In the experimental system, there are many step edges, each corresponding to a single interface. If the step edges are sufficiently separated

036129-1

from each other, they can be considered as independently growing interfaces. However, in general, there will always be an interaction, albeit weak, between two step edges and their motions will be coupled. Hence it is important to study persistence in models where the interface heights are coupled. Motivated by this observation, we study here the persistence in a simple model of coupled interfaces. While this model is, by no means, a true representative of the actual experimental situation, the advantage is that it represents a minimal model with coupling for which one can calculate the persistence properties analytically.

The paper is organized as follows. In Sec. II, we define the model precisely and state our main results. In Sec. III we consider the two limiting cases $d \to 0$ and $d = 1$ where our model reduces to two well known models. In Sec. IV we point out our strategy to compute the persistence exponents analytically. In Sec. V, we map the relevant stochastic process into a fractional Brownian motion (FBM) and then using a well known first-passage property of FBM, we calculate the persistence exponents analytically. Section VI considers a generalization to other types of interfaces. Section VII contains details of numerical results, and finally we conclude with a summary and outlook in Sec. VIII.

II. MODEL AND MAIN RESULTS

We consider two interfaces characterized by their heights $h_1(\mathbf{r}, t)$ and $h_2(\mathbf{r}, t)$ growing on a d-dimensional substrate according to the following evolution equations:

$$\frac{\partial h_1}{\partial t} = \nabla^2 h_1 + \eta_1(\mathbf{r}, t), \tag{1}$$

$$\frac{\partial h_2}{\partial t} = \nabla^2 h_2 + v(h_1(\mathbf{r}, t)) + \eta_2(\mathbf{r}, t), \tag{2}$$

where η_1 and η_2 represent the thermal Gaussian noises that are uncorrelated with each other, each has zero mean, and their correlators are given by

$$\langle \eta_1(\mathbf{r_1}, t_1) \eta_1(\mathbf{r_2}, t_2) \rangle = [4\pi a^2]^{-d/2} e^{-|\mathbf{r_1} - \mathbf{r_2}|^2/4a^2} \delta(t_1 - t_2), \tag{3}$$

$$\langle \eta_2(\mathbf{r_1}, t_1) \eta_2(\mathbf{r_2}, t_2) \rangle = [4\pi a^2]^{-d/2} e^{-|\mathbf{r_1} - \mathbf{r_2}|^2/4a^2} \delta(t_1 - t_2), \tag{4}$$

where a denotes the range of the correlator and serves as a short distance cutoff. For $d < 2$ where the interface roughens with time, the short distance cutoff plays no important role and one can safely take the limit $a \to 0$ and replace the correlators in Eqs. (3) and (4) by delta functions, i.e., $[4\pi a^2]^{-d/2} e^{-|\mathbf{r_1} - \mathbf{r_2}|^2/4a^2} \to \delta(\mathbf{r_1} - \mathbf{r_2})$. However, for $d > 2$, Eqs. (1) and (2) exhibit ultraviolet divergences and one needs to keep a finite a in order that $\langle h_1^2 \rangle$ and $\langle h_2^2 \rangle$ remain finite in the $t \to \infty$ limit.

The first interface h_1 evolves freely according to Eq. (1) which is precisely the celebrated Edwards-Wilkinson equation [29] in $(d+1)$ dimensions. On the other hand, the evo-

lution of the second interface h_2 in Eq. (2), in addition to having the Laplacian and the noise term, is coupled to the height h_1 of the first interface via the quenched random velocity $v(h_1(\mathbf{r}, t))$ which is also considered to be a Gaussian with the following moments:

$$\overline{v(h_1)} = 0, \tag{5}$$

$$\overline{v(h_1)v(h_1')} = \delta(h_1 - h_1'), \tag{6}$$

where $\overline{(\cdots)}$ denotes averages over the different realizations of the quenched velocity field $v(h_1)$.

We are interested in the following persistence probabilities at a fixed position $\mathbf{r}$ in space:

$$P_1(t, t_0) = \text{Prob}[h_1(\mathbf{r}, t') \neq h_1(\mathbf{r}, t_0)$$
$$\text{for all } t': t_0 < t' < t_0 + t], \tag{7}$$

$$P_2(t, t_0) = \text{Prob}[h_2(\mathbf{r}, t') \neq h_2(\mathbf{r}, t_0)$$
$$\text{for all } t': t_0 < t' < t_0 + t]. \tag{8}$$

The former represents the probability that the height $h_1(\mathbf{r}, t')$ of the first interface at a fixed point $\mathbf{r}$ in space does not return to its value at t_0 within the time interval $[t_0, t_0 + t]$. The latter represents the same probability for the second interface. The persistence probability $P_1(t, t_0)$ for the free Edwards-Wilkinson equation in Eq. (7) has been studied both analytically and numerically [20]. It is known that for $d < 2$, $P_1(t, t_0)$ has a power law decay for large t characterized by a nontrivial persistence exponent and the value of this exponent depends on whether the waiting time $t_0 = 0$ (no return to the initial condition) or $t_0 \to \infty$ (no return to a stationary configuration) [20],

$$P_1(t, t_0 = 0) \sim t^{-\theta_0^1}, \tag{9}$$

$$P_1(t, t_0 \to \infty) \sim t^{-\theta_s^1}, \tag{10}$$

where the subscripts 0 and s in the exponents refer respectively to the probabilities of no return to the initial condition ($t_0 = 0$) and no return to a stationary configuration ($t_0 \to \infty$). The superscript 1 refers to the first interface. It turns out that the exponent θ_0^1 is hard to determine analytically and is known only numerically. For example, $\theta_0^1 = 1.55 \pm 0.02$ in $(1+1)$ dimensions [20]. On the other hand, the exponent θ_s^1 was computed analytically [20] by mapping the process h_1 in the $t_0 \to \infty$ limit to a FBM and then using a known first-passage result of FBM. For all $d < 2$, one gets [20]

$$\theta_s^1 = (2 + d)/4. \tag{11}$$

For $d = 1$ this gives $\theta_s^1 = 3/4$, a result that was recently verified experimentally in a system of fluctuating $(1+1)$-dimensional steps on Si-Al surfaces [28]. For $d > 2$, we show that the persistence $P_1(t, t_0 \to \infty)$ decays faster than a power law for large t, stretched exponentially for $2 < d < 4$ and exponentially for $d > 4$.

In this paper, we study the persistence probability $P_2(t, t_0)$ of the second interface in Eq. (2). As in the case of the first interface, we find that for large interval t

$$P_2(t, t_0 = 0) \sim t^{-\theta_0^2}, \tag{12}$$

$$P_2(t, t_0 \to \infty) \sim t^{-\theta_s^2}, \tag{13}$$

where the superscript 2 in the exponents refer to the second interface. While we were not able to compute the exponent θ_0^2, we calculated the exponent θ_s^2 analytically for all $d < 2$,

$$\theta_s^2 = (2 - d)/8. \tag{14}$$

For $d > 2$, we argue that the persistence $P_2(t, t_0 \to \infty)$ tends to a constant as $t \to \infty$.

Furthermore, we were able to generalize this result to the case when the second interface evolves by Eq. (2) but the first interface is any generic growing surface, evolving not neccesarily by the Edwards-Wilkinson equation. For example, the first interface may evolve by the Kardar-Parisi-Zhang (KPZ) equation [30]. In general, this first interface will be characterized by a growth exponent β_1 and a dynamical exponent z_1 defined via the scaling form of the second moment of the height differences between two points in space, $\langle [h_1(\mathbf{r}_1, \tau_1) - h_1(\mathbf{r}_2, \tau_2)]^2 \rangle \approx |\tau_2 - \tau_1|^{2\beta_1} f(|\mathbf{r}_1 - \mathbf{r}_2|^{z_1}/|\tau_2 - \tau_1|)$. For example, for the $(1+1)$-dimensional Edwards-Wilkinson equation one has $\beta_1 = 1/4$ and $z_1 = 2$, whereas for the $(1+1)$-dimensional KPZ equation one has $\beta_1 = 1/3$ and $z_1 = 3/2$ [31]. Our main result is to show that

$$\theta_s^2 = \beta_1/2. \tag{15}$$

In particular, Eq. (15) predicts that in $(1+1)$ dimensions, if the first interface evolves via the KPZ equation, $\theta_s^2 = 1/6$. In Sec. VII we show that the numerical results are consistent with this theoretical prediction.

III. LIMITING CASES

In this section we show that in the two limiting cases $d \to 0$ and $d = 1$, our model defined by Eqs. (1) and (2) reduce respectively to two well studied models.

A. Limit $d \to 0$

In the limit $d \to 0$, there is no "space" variable in the problem. Thus the Laplacian terms on the right-hand side of Eqs. (1) and (2) drop out and also the noise variables no longer have any $\mathbf{r}$ dependence. Interpreting $h_1 = x$ and $h_2 = y$ as the coordinates (x, y) of a single particle on a two-dimensional plane, Eqs. (1) and (2) reduce to

$$\frac{dx}{dt} = \eta_1(t), \tag{16}$$

$$\frac{dy}{dt} = v(x(t)) + \eta_2(t). \tag{17}$$

These equations represent precisely the Matheron–de Marsily (MdM) model [15] where a single Brownian particle moves in a two-dimensional plane in the presence of a transverse quenched random velocity field $v(x)$. This model was originally introduced to study the hydrodynamic dispersion
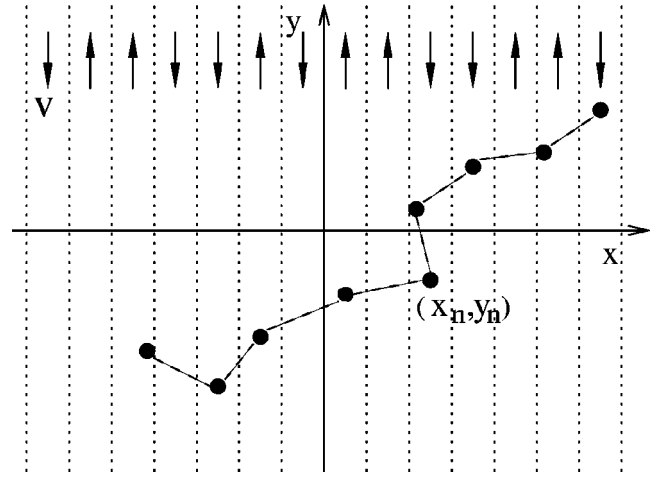

FIG. 1. A Rouse chain in a random layered velocity field in two dimensions.

of a tracer particle in porous rocks [15]. While the transport properties in this model were well understood [32–34], the study of persistence properties in this model are relatively recent [12,13]. The persistence probabilities defined in Eqs. (7) and (8) reduce, in this context, to the following probabilities:

$$P_1(t, t_0) = \text{Prob}[x(t') \neq x(t_0) \text{ for all } t' : t_0 < t' < t_0 + t], \tag{18}$$

$$P_2(t, t_0) = \text{Prob}[y(t') \neq y(t_0) \text{ for all } t' : t_0 < t' < t_0 + t]. \tag{19}$$

The first probability $P_1(t, t_0)$ is simply the probability of no return to its initial position of a one-dimensional Brownian motion and hence $P_1(t, t_0) \sim t^{-\theta_{0,s}^1}$ where $\theta_0^1 = \theta_s^1 = 1/2$. The probability $P_2(t, t_0)$ associated with the y coordinate is nontrivial. Based on heuristic arguments and numerical simulations, Redner showed that $\theta_0^2 = 1/4$ [12]. More recently, it was proved analytically that $\theta_0^2 = \theta_s^2 = 1/4$ by mapping the stochastic process $y(t)$ to a FBM and then using a known first-passage property of the latter [13]. The result $\theta_s^2 = 1/4$ is thus a special case of our result in Eq. (14) of the interface model in the limit $d \to 0$. Incidentally, the exponent $\theta_0^2 = 1/4$ happens to be generic for a class of transverse velocity fields and occurs even when the tranverse velocity field is a deterministic but an odd function of x, i.e., $v(x) = -v(-x)$ [35,36,14].

B. Case $d = 1$

Let us consider a Rouse polymer chain embedded in a two-dimensional plane. The chain consists of beads connected by harmonic springs [16]. In addition, the chain is advected by a random layered velocity field as shown in Fig. 1. Let $[x_n(t), y_n(t)]$ denote the coordinates of the nth bead at time t which evolve with time according to the following equations of motion:

$$\frac{dx_n}{dt} = \Gamma(x_{n+1} + x_{n-1} - 2x_n) + \eta_1(n, t), \tag{20}$$

$$\frac{dy_n}{dt} = \Gamma(y_{n+1} + y_{n-1} - 2y_n) + v(x_n(t)) + \eta_2(n,t), \quad (21)$$

where Γ denotes the strength of the harmonic interaction between nearest neighbor beads, $\eta_1(n,t)$ and $\eta_2(n,t)$ represent the Gaussian thermal noises along the x and y directions, respectively, and are delta correlated. The velocity field $v(x)$ is a random quenched function of x taken to be a Gaussian with the following moments:

$$\overline{v(x)} = 0, \quad (22)$$

$$\overline{v(x)v(x')} = \delta(x - x'). \quad (23)$$

For a finite chain with N beads, Eqs. (20) and (21) are valid only for the $(N-2)$ interior beads. The two boundary beads will have slightly different equations of motion. However, we will only focus here on an infinitely large chain ($N \rightarrow \infty$) so that the system is translationally invariant along the length of the chain and the boundary conditions are irrelevant.

Note that in the absence of the harmonic interaction term, i.e., when $\Gamma=0$, this model reduces precisely to a single particle MdM model discussed in the previous subsection. In presence of the harmonic interaction, the transport properties in this model have been studied recently [17–19]. However, the persistence properties for $\Gamma \neq 0$ are nontrivial. One can define the following persistence probabilities:

$$P_1(t,t_0) = \text{Prob}[x_n(t') \neq x_n(t_0) \quad \text{for all } t':t_0 < t' < t_0 + t], \quad (24)$$

$$P_2(t,t_0) = \text{Prob}[y_n(t') \neq y_n(t_0) \quad \text{for all } t':t_0 < t' < t_0 + t], \quad (25)$$

where the former represents the probability that the x coordinate of a bead (say the nth bead) does not return to its position at time t_0 within the time interval $[t_0, t_0+t]$, while the latter represents the same probability for the y coordinate of the same bead. For an infinite chain, the system is translationally invariant along the length of the chain and hence these persistence probabilities do not depend on the bead label n.

Since we are interested in the late time properties, one can conveniently replace the harmonic interaction term in Eqs. (20) and (21) by a continuous Laplacian operator,

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2} + \eta_x(s,t), \quad (26)$$

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial s^2} + v(x(s,t)) + \eta_y(s,t), \quad (27)$$

where s denotes the distance along the chain and we have rescaled the time to set the coefficient in front of the Laplacian to be unity. Interpreting $x \equiv h_1$ and $y \equiv h_2$, Eq. (26) and (27) reduce precisely to the $d=1$ version of our model defined in Eqs. (1) and (2). Thus substituting $d=1$ in our results for the persistence exponents in Eqs. (11) and (14), we obtain the persistence exponents for the polymer problem. In par-

ticular, for the probability of no return to a stationary configuration [$t_0 \rightarrow \infty$ limit in Eqs. (24) and (25)] of the polymer, we have for large interval t,

$$P_1(t, t_0 \rightarrow \infty) \sim t^{-3/4}, \quad (28)$$

$$P_2(t, t_0 \rightarrow \infty) \sim t^{-1/8}, \quad (29)$$

where the exponents $3/4$ and $1/8$ are obtained by substituting $d=1$ in Eqs. (11) and (14), respectively. Interestingly, for $\Gamma=0$, i.e., in the single particle MdM model, the corresponding probabilities decay as $P_1(t, t_0 \rightarrow \infty) \sim t^{-1/2}$ and $P_2(t, t_0 \rightarrow \infty) \sim t^{-1/4}$, as discussed in the previous subsection. Thus switching on the harmonic interaction strength Γ has opposite effects on P_1 and P_2. While the probability $P_1(t, t_0 \rightarrow \infty)$ decays faster as $\sim t^{-3/4}$ (compared to $\sim t^{-1/2}$ when $\Gamma=0$) in presence of interaction Γ, the probability $P_2(t, t_0 \rightarrow \infty)$ decays slower as $\sim t^{-1/8}$ when Γ is switched on (compared to $\sim t^{-1/4}$ when $\Gamma=0$).

IV. STRATEGY TO COMPUTE THE PERSISTENCE EXPONENTS

The stochastic process $h_1(\mathbf{r},t)$ in Eq. (1) for a fixed $\mathbf{r}$, as a function of time t, is Gaussian albeit non-Markovian. The non-Markovian property arises due to the Laplacian term that generates interaction between heights at two different space points. The process $h_2(\mathbf{r},t)$ in Eq. (2) for fixed $\mathbf{r}$ is similarly non-Markovian and moreover it is non-Gaussian due to the quenched velocity field in Eq. (2). Analytical calculation of the persistence exponent is known to be very hard for a non-Markovian process even if the process is Gaussian [4]. For a non-Gaussian process it is even harder in general. However, in certain cases, it may be possible to map the relevant stochastic process into a FBM and then one can use a known first-passage property of FBM to calculate the persistence exponent analytically. This strategy has been successful in the past to compute analytically the persistence exponents in a number of problems even though the relevant processes were non-Gaussian and/or non-Markovian [13,20–22,26,27]. In particular, this strategy was used in Ref. [20] to calculate the exponent $\theta_s^1 = (2+d)/4$ in Eq. (11) for the first interface evolving freely with a Edwards-Wilkinson dynamics. Here we exploit the same strategy to compute the exponent $\theta_s^2 = (2-d)/8$ in Eq. (14) for the second interface.

Before proceeding further, it is then useful at this point to summarize the definition and the known first-passage property of a FBM. A stochastic process $X(t)$ (with zero mean $E[X(t)]=0$) is called a FBM if its incremental two-time correlation function $C(t_1,t_2) = E[(X(t_1)-X(t_2))^2]$ is (i) *stationary*, i.e., depends only on the time difference $|t_2-t_1|$, and (ii) grows asymptotically as a power law,

$$C(t_1,t_2) \sim |t_2 - t_1|^{2H}, \quad |t_2 - t_1| \gg 1. \quad (30)$$

The parameter $0<H<1$ is called the Hurst exponent that chracterizes the FBM [37] and $E[\cdots]$ denotes the expectation value over all realizations of the process $X(t)$. For example, an ordinary Brownian motion corresponds to a FBM with $H=1/2$. The zero crossing properties of a FBM has been

studied extensively in the past [38–40,20]. The particular property that is useful for our purpose is the fact that the probability that a FBM does not cross zero up to time t decays as a power law at late times, $P(t) \sim t^{-\theta}$ with $\theta = 1 - H$ [40,20]. Note that this result $\theta = 1 - H$ holds for any zero mean process that satisfies the defining characteristics (i) and (ii) above of a FBM. In particular, it holds even if the process is non-Gaussian and/or non-Markovian as long it satisfies (i) and (ii) above. A simple proof of this result is given in Ref. [20]. Our strategy in the next section would be to first show that indeed for $d < 2$, the processes $h_1(\mathbf{r}, t)$ and $h_2(\mathbf{r}, t)$ in Eqs. (1) and (2), for fixed $\mathbf{r}$ and in the limit $t_0 \to \infty$, do satisfy the characteristics (i) and (ii) of a FBM with Hurst exponents $H_1 = (2 - d)/4$ and $H_2 = (6 + d)/8$, respectively, and then use the powerful result $\theta = 1 - H$ stated above to prove that $\theta_s^1 = (2 + d)/4$ and $\theta_s^2 = (2 - d)/8$. For the free interface h_1 in Eq. (1) this mapping to the FBM was already done in Ref. [20], but we will include it Sec. V A for completeness. The corresponding calculations for h_2 will be detailed in Sec. V B.

V. TWO-TIME CORRELATION FUNCTIONS

In order to make use of the first-passage property of FBM mentioned in Sec. IV, we need to calculate the incremental correlation functions,

$$C_1(t_1, t_2, t_0) = E[(h_1(\mathbf{r}, t_0 + t_1) - h_1(\mathbf{r}, t_0 + t_2))^2], \quad (31)$$

$$C_2(t_1, t_2, t_0) = E[(h_2(\mathbf{r}, t_0 + t_1) - h_2(\mathbf{r}, t_0 + t_2))^2], \quad (32)$$

where $E[\cdots]$ denotes an average over both thermal noises as well as the disorder. Note that while for the first interface h_1 the expectation $E[\cdots]$ is only over the thermal noise, for the second interface it includes averages over both the thermal noise as well as the disorder. In Sec. V A, we calculate $C_1(t_1, t_2, t_0)$ which is rather straightforward. The computation of $C_2(t_1, t_2, t_0)$ is more involved and is detailed in Sec. V B.

A. Incremental correlation function for the first interface

Since Eq. (1) is linear one can solve it by the standard Fourier transform technique. We define the Fourier transform $\tilde{h}_1(\mathbf{k}, t) = \int d^d \mathbf{r} \, h_1(\mathbf{r}, t) \exp(i\mathbf{k} \cdot \mathbf{r})$. Taking the Fourier transform of Eq. (1) one gets

$$\tilde{h}_1(\mathbf{k}, t) = e^{-k^2 t} \int_0^t e^{k^2 t'} \tilde{\eta}_1(\mathbf{k}, t') dt', \quad (33)$$

where we have assumed that the system starts from a flat initial condition $h_1(\mathbf{r}, t=0) = 0$. Using the noise correlator in Eq. (3) one can then easily show that

$$\langle \tilde{h}_1(\mathbf{k}, t_1) \tilde{h}_1(-\mathbf{k}, t_2) \rangle = \frac{1}{2k^2} [e^{-k^2|t_2 - t_1|} - e^{-k^2(t_1 + t_2)}] e^{-k^2 a^2}. \quad (34)$$

The autocorrelation function in real space can then be obtained as

$$A_1(t_1, t_2) = \langle h_1(\mathbf{r}, t_1) h_1(\mathbf{r}, t_2) \rangle$$

$$= \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{e^{-k^2 a^2}}{2k^2} [-(1 - e^{-k^2|t_2 - t_1|}) + (1 - e^{-k^2(t_1 + t_2)})]. \quad (35)$$

The right-hand side of Eq. (35) is a sum of two integrals each of which is convergent and can be easily done in closed form and one gets [20]

$$A_1(t_1, t_2) = \frac{a_0}{(2 - d)} [(t_1 + t_2 + a^2)^{1 - d/2} - (|t_2 - t_1| + a^2)^{1 - d/2}], \quad (36)$$

where $a_0 = (4\pi)^{-d/2}$. Thus, as $t \to \infty$, $A_1(t, t) = \langle h_1^2(\mathbf{r}, t) \rangle \sim t^{1 - d/2} = t^{2\beta_1}$ for $d < 2$. One the other hand, for $d > 2$, $A_1(t, t) \to a_0/[(d-2)a^{d-2}]$ as $t \to \infty$. Thus while for $d > 2$ the surface becomes smooth in the stationary state, for $d < 2$ the surface roughens with time and the fluctuations grow as a power law in an infinite system,

$$\langle h_1^2(\mathbf{r}, t) \rangle \sim t^{2\beta_1}, \quad \text{where } \beta_1 = (2 - d)/4. \quad (37)$$

The incremental two-time correlation function defined in Eq. (31) can then be written as

$$C_1(t_1, t_2, t_0) = \langle [h_1(\mathbf{r}, t_1 + t_0) - h_1(\mathbf{r}, t_2 + t_0)]^2 \rangle$$

$$= A_1(t_1 + t_0, t_1 + t_0) + A_1(t_2 + t_0, t_2 + t_0) - 2A_1(t_1 + t_0, t_2 + t_0). \quad (38)$$

Using Eq. (36) and taking the limit $t_0 \to \infty$ one finds that for $d < 2$

$$C_1(t_1, t_2, t_0 \to \infty) = \frac{a_0}{(2 - d)} [2(|t_2 - t_1| + a^2)^{1 - d/2} - 2a^{2 - d}]$$

$$\sim |t_2 - t_1|^{1 - d/2} \quad \text{for } |t_2 - t_1| \gg a^2. \quad (39)$$

Comparing with the defining property of the FBM in Eq. (30) we see that for $d < 2$ and in the limit $t_0 \to \infty$, i.e., when one observes the stochastic process $h_1(\mathbf{r}, t)$ at a fixed point $\mathbf{r}$ in space as a function of time after a waiting time $t_0 \to \infty$, the process is a FBM with a Hurst exponent $H_1 = (2 - d)/4$. This then proves that the persistence $P_1(t, t_0 \to \infty) \sim t^{-\theta_s^1}$ as $t \to \infty$ where

$$\theta_s^1 = 1 - H_1 = (2 + d)/4. \quad (40)$$

In the limit $d \to 0$, Eq. (40) gives $\theta_s^1 = 1/2$ (the classical Brownian motion result) and for $d = 1$, one gets $\theta_s^1 = 3/4$, a result that was predicted in Ref. [20] and was subsequently verified experimentally [28].

We now turn to the persistence probability $P_1(t, t_0 \to \infty)$ for $d > 2$. For $d > 2$, the physics of the process is rather different. The fluctutations of height h_1 no longer grow at late times, but rather saturate to a constant, $\langle h_1^2(\mathbf{r}, t) \rangle = A_1(t, t) \to a_0/[(d-2)a^{d-2}]$ as $t \to \infty$. Thus, for $d > 2$, the cutoff $a > 0$ is essential. Besides, the relevant process $h_1(\mathbf{r}, t)$, at a fixed $\mathbf{r}$ but as a function of time, no longer satisfies the properties of a FBM. Thus one can no longer use the first-passage property of FBM to compute the persistence

$P_1(t, t_0 \to \infty)$. To make progress, we note that for $d > 2$, it follows from Eq. (36) that in the limit when both t_1 and t_2 are large but their difference $|t_2 - t_1| = t$ kept fixed, the autocorrelation function becomes stationary, i.e., only a function of $t = |t_2 - t_1|$, and decays as a power law

$$A_1(t_1, t_2) \approx \frac{a_0}{(d-2)} \frac{1}{(|t_2 - t_1| + a^2)^{d/2-1}}. \tag{41}$$

Thus, in this limit, h_1 is a Gaussian stationary process with an autocorrelator $A_1(|t_2 - t_1| = t) \sim t^{-(d/2-1)}$. Interestingly, exactly the same Gaussian stationary process also represents the total magnetization of a manifold in nonequilibrium critical dynamics within the mean field theory [41]. In general, the calculation of persistence of a Gaussian stationary process with an algebraically decaying correlator is nontrivial. However, it was pointed out in Ref. [41] that one can use a powerful theorem due to Newell and Rosenblatt [42] to obtain useful bounds on the persistence property of such a Gaussian process. The Newell-Rosenblatt theorem states that if the stationary autocorrelator of a Gaussian process decays algebraically as $A(t) \sim t^{-\alpha}$ for large time difference $|t_2 - t_1| = t$, then the persistence $P(t)$ (i.e., the probability of no zero crossing over a time interval t) has the following asymptotic behaviors:

$$P(t) \sim \exp[-K_1 t] \quad \text{for } \alpha > 1,$$

$$\exp[-K_2 t^\alpha \ln t] \leq P(t) \leq \exp[-K_3 t^\alpha] \quad \text{for } 0 < \alpha < 1, \tag{42}$$

where K_1, K_2, and K_3 are some constants. In the borderline case $\alpha = 1$, one has additional logarithmic correction. Applying this theorem to our interface problem upon identifying $\alpha = d/2 - 1 > 0$, we conclude that for large t

$$P_1(t, t_0 \to \infty) \sim \exp[-K_1 t] \quad \text{for } d > 4,$$

$$\exp[-K_2 t^{d/2-1} \ln t] \leq P_1(t, t_0 \to \infty) \leq \exp[-K_3 t^{d/2-1}] \quad \text{for } 2 < d < 4. \tag{43}$$

Thus the persistence decays exponentially at late times for $d > 4$ and stretched exponentially for $2 < d < 4$. This is to be contrasted with the power law decay $P_1(t, t_0 \to \infty) \sim t^{-\theta_s^1}$ for $d < 2$.

B. Incremental correlation function for the second interface

In this subsection we show that in the limit $t_0 \to \infty$ and for $d < 2$, even h_2 is a FBM process and one can calculate exactly the corresponding Hurst exponent H_2 and hence the persistence exponent $\theta_s^2 = 1 - H_2$. Taking the Fourier transform of Eq. (2) one gets

$$\tilde{h}_2(\mathbf{k}, t) = e^{-k^2 t} \int_0^t dt' e^{k^2 t'} \left[\tilde{\eta}_2(\mathbf{k}, t') \right.$$

$$\left. + \int e^{i\mathbf{k} \cdot \mathbf{r}'} v(h_1(\mathbf{r}', t')) d^d \mathbf{r}' \right]. \tag{44}$$

Alternatively in real space, one can write

$$h_2(\mathbf{r}, t) = \int_0^t dt' \int d^d \mathbf{r}' \frac{e^{-(\mathbf{r} - \mathbf{r}')^2/4(t-t')}}{[4\pi(t-t')]^{d/2}} [\eta_2(\mathbf{r}', t') + v(h_1(\mathbf{r}', t'))]. \tag{45}$$

Then the autocorelation function, averaged over both the thermal noise and the disorder, is given by

$$A_2(t_1, t_2) = E[h_2(\mathbf{r}, t_1) h_2(\mathbf{r}, t_2)] = \overline{\langle h_2(\mathbf{r}, t_1) h_2(\mathbf{r}, t_2) \rangle}$$

$$= I_1(t_1, t_2) + I_2(t_1, t_2), \tag{46}$$

where $I_1(t_1, t_2)$ counts the contribution due to the noise-noise correlator $\langle \eta_2 \eta_2 \rangle$ and has the same expression as the autocorrelation function of the first interface,

$$I_1(t_1, t_2) = A_1(t_1, t_2) = \frac{a_0}{(2-d)} [(t_1 + t_2 + a^2)^{1-d/2}$$

$$- (|t_2 - t_1| + a^2)^{1-d/2}]. \tag{47}$$

On the other hand $I_2(t_1, t_2)$ counts the contribution due to the random velocity term and is given by

$$I_2(t_1, t_2) = \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \int d^d \mathbf{r}_1' \frac{e^{-(\mathbf{r} - \mathbf{r}_1')^2/4(t_1 - \tau_1)}}{[4\pi(t_1 - \tau_1)]^{d/2}}$$

$$\times \int d^d \mathbf{r}_2' \frac{e^{-(\mathbf{r} - \mathbf{r}_2')^2/4(t_2 - \tau_2)}}{[4\pi(t_2 - \tau_2)]^{d/2}} \overline{\langle v \cdot v \rangle}, \tag{48}$$

where

$$\overline{\langle v \cdot v \rangle} = \overline{\langle v[h_1(\mathbf{r}_1', \tau_1)] v[h_1(\mathbf{r}_2', \tau_2)] \rangle}$$

$$= \langle \delta(h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dq}{2\pi} \langle e^{iq[h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]} \rangle. \tag{49}$$

In Eq. (49) we have first performed the disorder average which gives the expression on the second line. Next we used an integral representation of the delta function in the third line. The thermal average over the delta function can be done by noting that for a Gaussian process h, $\langle \delta(h) \rangle = \int_{-\infty}^{\infty} [dq/(2\pi)] \langle e^{iqh} \rangle = \int_{-\infty}^{\infty} [dq/(2\pi)] e^{-q^2 \langle h^2 \rangle/2} = 1/\sqrt{2\pi \langle h^2 \rangle}$; in our case h_1 being a Gaussian process, $[h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]$ is also Gaussian. This gives

$$\overline{\langle v \cdot v \rangle} = \frac{1}{\sqrt{2\pi \langle [h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]^2 \rangle}}. \tag{50}$$

Using Eq. (34) one can easily compute the following two-time correlation function:

$$\langle h_1(\mathbf{r}_1', \tau_1) h_1(\mathbf{r}_2', \tau_2) \rangle = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \langle \tilde{h}_1(\mathbf{k}, t_1) \tilde{h}_1(-\mathbf{k}, t_2) \rangle e^{i\mathbf{k} \cdot (\mathbf{r}_1' - \mathbf{r}_2')}$$

$$= \frac{1}{2} \int_{|\tau_1 - \tau_2| + a^2}^{\tau_1 + \tau_2 + a^2} d\tau \frac{e^{-(\mathbf{r}_1' - \mathbf{r}_2')^2/4\tau}}{(4\pi\tau)^{d/2}}. \tag{51}$$

For $d < 2$, one can safely set the cutoff $a \to 0$ for simplicity and using Eq. (51) one gets

$$\langle [h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]^2 \rangle = \frac{a_0}{(2-d)} \Bigg[(2\tau_1)^{1-d/2} + (2\tau_2)^{1-d/2}$$

$$- (2-d) \int_{|\tau_1 - \tau_2|}^{\tau_1 + \tau_2} d\tau \tau^{-d/2}$$

$$\times e^{-(\mathbf{r}_1' - \mathbf{r}_2')^2/4\tau} \Bigg]. \tag{52}$$

Equations (52) and (50) thus fully specify $I_2(t_1, t_2)$ in Eq. (48) which can be further simplified by making the change of variables: $\mathbf{r}_1' - \mathbf{r} = \mathbf{u}$ and $\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{z}$. The integral over $\mathbf{u}$ is a Gaussian that can be easily performed. After a few steps of elementary algebra we get

$$I_2(t_1, t_2) = \sqrt{2(2-d)}(4\pi)^{(d-2)/4} \int d^d\mathbf{z} \frac{e^{-\mathbf{z}^2/4(t_1+t_2-\tau_1-\tau_2)}}{[4\pi(t_1 + t_2 - \tau_1 - \tau_2)]^{d/2}}$$

$$\times \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \Bigg[(2\tau_1)^{1-d/2} + (2\tau_2)^{1-d/2} - (2-d)$$

$$\times \int_{|\tau_1 - \tau_2|}^{\tau_1 + \tau_2} d\tau \tau^{-d/2} e^{-\mathbf{z}^2/4\tau} \Bigg]^{-1/2}. \tag{53}$$

Note that when $t_1 = t_2 = t$ a simple power counting in Eq. (53) shows that $I_2(t, t) \sim t^{(d+6)/4}$ for large t. On the other hand, it follows from Eq. (47) that $I_1(t, t) \sim t^{(2-d)/2}$ for large t. Thus $I_2(t, t)$ grows faster than $I_1(t, t)$ for large t. Thus it follows from Eq. (46) that

$$E[h_2^2(\mathbf{r}, t)] = A_2(t, t) \sim t^{2\beta_2}, \quad \text{where } \beta_2 = (6+d)/8. \tag{54}$$

In particular for $d=1$, the result $\beta_2 = 7/8$ agrees with that of Ref. [19] derived in the context of a Rouse chain in a random tranverse velocity field.

The incremental correlation function defined in Eq. (32) can then be written in terms of the autocorrelation function,

$$C_2(t_1, t_2, t_0) = E\left[(h_2(\mathbf{r}, t_1 + t_0) - h_2(\mathbf{r}, t_2 + t_0))^2 \right]$$

$$= A_2(t_1 + t_0, t_1 + t_0) + A_2(t_2 + t_0, t_2 + t_0)$$

$$- 2A_2(t_1 + t_0, t_2 + t_0)$$

$$= C_1(t_1, t_2, t_0) + C_2^{I_2}, \tag{55}$$

where in going from the second to the third line in Eq. (55) we have used Eq. (46), the fact that $C_1(t_1, t_2, t_0) = I_1(t_1 + t_0, t_1 + t_0) + I_1(t_2 + t_0, t_2 + t_0) - 2I_1(t_1 + t_0, t_2 + t_0)$ and the contribution $C_2^{I_2}$ coming from the I_2 terms is defined as

$$C_2^{I_2} = I_2(t_1 + t_0, t_1 + t_0) + I_2(t_2 + t_0, t_2 + t_0) - 2I_2(t_1 + t_0, t_2 + t_0). \tag{56}$$

A close inspection of Eq. (53) reveals that for large t_1 and t_2 the dominant contribution to the integral comes from the region where $(t_1 + t_2 - \tau_1 - \tau_2) \to 0$. To capture the leading behavior of $I_2(t_1, t_2)$ for large t_1 and t_2 it is then sufficient to make the replacement $e^{-\mathbf{z}^2/4(t_1+t_2-\tau_1-\tau_2)}/[4\pi(t_1 + t_2 - \tau_1 - \tau_2)]^{d/2} \to \delta(\mathbf{z})$ in Eq. (53). Thus to leading order for large t_1 and t_2 one gets

$$I_2(t_1, t_2) \approx \sqrt{2(2-d)}(4\pi)^{(d-2)/4} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 [(2\tau_1)^{1-d/2}$$

$$+ (2\tau_2)^{1-d/2} - 2(\tau_1 + \tau_2)^{1-d/2} + 2|\tau_1 - \tau_2|^{1-d/2}]^{-1/2}. \tag{57}$$

For large t_0, all the arguments of I_2 on the right-hand side rhs of Eq. (56) are large even though t_1 and t_2 may not be large. Hence one can use the asymptotic expression of I_2 in Eq. (57). Substituting Eq. (57) in Eq. (56) and rearranging the domains of integration one finds for large t_0

$$C_2^{I_2} \approx \sqrt{2(2-d)}(4\pi)^{(d-2)/4} \int_{t_1+t_0}^{t_2+t_0} d\tau_1 \int_{t_1+t_0}^{t_2+t_0} d\tau_2 [(2\tau_1)^{1-d/2}$$

$$+ (2\tau_2)^{1-d/2} - 2(\tau_1 + \tau_2)^{1-d/2} + 2|\tau_1 - \tau_2|^{1-d/2}]^{-1/2}. \tag{58}$$

Making a further change of variables $x_1 = \tau_1 - (t_1 + t_0)$ and $x_2 = \tau_2 - (t_2 + t_0)$ one finds that to leading order for large t_0 the rhs of Eq. (58) becomes independent of t_0 and depends only on $|t_2 - t_1|$,

$$C_2^{I_2} \approx \sqrt{(2-d)}(4\pi)^{(d-2)/4} \int_0^{|t_2-t_1|} dx_1 \int_0^{|t_2-t_1|} dx_2 |x_1 - x_2|^{(d-2)/4}. \tag{59}$$

Performing the double integral in Eq. (59) one finally gets

$$C_2^{I_2} \approx \frac{32\sqrt{(2-d)}}{(d+2)(d+6)}(4\pi)^{(d-2)/4}|t_2 - t_1|^{(6+d)/4}. \tag{60}$$

Substituting the results from Eqs. (39) and (60) on the rhs of Eq. (55) one gets for large t_0

$$C_2(t_1, t_2, t_0 \to \infty) \approx b_1|t_2 - t_1|^{1-d/2} + b_2|t_2 - t_1|^{(6+d)/4}, \tag{61}$$

where b_1 and b_2 are two constants that can be read off Eqs. (39) and (60), respectively. Since the exponent $(6+d)/4 > (1-d/2)$ one gets for large $|t_2 - t_1|$

$$C_2(t_1, t_2, t_0 \to \infty) \sim |t_2 - t_1|^{(6+d)/4}. \tag{62}$$

Comparing with Eq. (30) one thus finds that for large t_0 and $d < 2$, the process h_2 is also a FBM with

$$H_2 = (6+d)/8, \tag{63}$$

$$\theta_s^2 = 1 - H_2 = (2-d)/8 = \beta_1/2. \tag{64}$$

In the limit $d \to 0$, one thus recovers the result $\theta_s^2 = 1/4$ for a single particle MdM model [12,13]. For $d=1$, we get $\theta_s^2 = 1/8$, the exact persistence exponent for the Rouse chain in a transverse velocity field.

The above results are valid for $d < 2$. For $d > 2$, one needs to keep the cutoff a finite. Carrying out a similar analysis as in the $d < 2$ case but keeping a finite, one can show that in the limit $t_0 \to \infty$, the incremental correlation function has the following asymptotic behavior for all $d > 2$:

$$C_2(t_1, t_2, t_0 \to \infty) \sim |t_2 - t_1|^2 \quad \text{for } |t_2 - t_1| \gg a^2. \quad (65)$$

Thus for $d > 2$, h_2 is a FBM with $H_2 = 1$ which indicates that $\theta_s^2 = 0$ for all $d > 2$. This means that the persistence $P_2(t, t_0 \to \infty)$ tends to a constant at large time t for all $d > 2$. Indeed, Eq. (65) indicates that the surface h_2 grows ballistically at late times and with a finite probability h_2 does not return to its starting position at t_0 over the time interval $[t_0, t + t_0]$. In combination with the result in Eq. (64), one thus concludes that the exponent $\theta_s^2 = (2 - d)/8$ for $d < 2$ tends to 0 as $d \to 2^-$ and then sticks to $\theta_s^2 = 0$ for all $d > 2$. Note that this behavior of $P_2(t, t_0 \to \infty)$ for $d > 2$ is quite opposite to the corresponding $d > 2$ behavior of the first interface h_1 for which $P_1(t, t_0 \to \infty)$ decays faster than a power law at large t as was shown in the previous subsection.

VI. GENERALIZATION TO OTHER GROWING INTERFACES

In this section, we consider a generalization of our model of coupled interfaces. In this generalized version, while the second interface height h_2 still evolves via Eq. (2), the first interface height h_1 may correspond to any generically growing interface, not necessarily evolving via the Edwards-Wilkinson equation (1). For example, h_1 may evolve via the KPZ equation [30],

$$\frac{\partial h_1}{\partial t} = \nabla^2 h_1 + \lambda (\nabla h_1)^2 + \eta_1(\mathbf{r}, t). \quad (66)$$

In general, we will consider a generically growing interface h_1 characterized by the following dynamical scaling of its space-time correlation function [31],

$$\langle [h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]^2 \rangle \approx |\tau_2 - \tau_1|^{2\beta_1} f\left(\frac{|\mathbf{r}_1' - \mathbf{r}_2'|^{z_1}}{|\tau_2 - \tau_1|}\right), \quad (67)$$

where $\beta_1 > 0$ is the growth exponent, z_1 is the dynamical exponent, and $f(y)$ is the dynamical scaling function which approaches a constant as $y \to 0$, $f(0) = C$ and decays for large y. For example, for the $(1+1)$-dimensional Edwards-Wilkinson equation, $\beta_1 = 1/4$ and $z_1 = 2$, whereas for the $(1+1)$-dimensional KPZ equation, $\beta_1 = 1/3$ and $z_1 = 3/2$ [31]. Note that for a $(1+1)$-dimensional KPZ equation, one loses the symmetry $h \to -h$. Hence the persistence exponent θ_s^1 associated with h_1 will be different depending on whether the process h_1 stays above its mean value or below its mean value [21,26]. In this paper, we focus only on the second interface h_2 for which there is only one persistence exponent θ_s^2. The goal of this section is to show that quite generically $\theta_s^2 = \beta_1/2$.

We follow a similar route as in Sec. V B and start with the calculation of the autocorrelation function of the h_2. All the steps from Eqs. (44) and (49) remain unchanged since one does not use any information about h_1 until Eq. (49). One uses the specific information about h_1 in evaluating the average $\overline{\langle v \cdot v \rangle}$ in Eq. (50). In Sec. V B, the process h_1 is Gaussian at all times since it evolves via the linear Edwards-

Wilkinson equation (1). This fact that h_1 is Gaussian was used explicitly in evaluating the thermal average $\langle \exp\{iq[h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]\}\rangle$ in Eq. (49) which led to the result in Eq. (50). For a generic non-Gaussian interface h_1 one can use this step to evaluate the thermal average in Eq. (49). To make progress, let us first denote $h = [h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]$. Then Eq. (49) gives

$$\overline{\langle v \cdot v \rangle} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \langle e^{iqh} \rangle = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \int dh \, e^{iqh} P(h) \quad (68)$$

where $P(h)$ is the normalized probability distribution of the variable h. For a generic interface, one expects the normalized distribution to have the scaling form, $P(h) = 1/\sqrt{\langle h^2 \rangle} F(h/\sqrt{\langle h^2 \rangle})$. Substituting this scaling form in Eq. (68) and rescaling, one finds the following scaling:

$$\overline{\langle v \cdot v \rangle} \sim \frac{1}{\sqrt{\langle h^2 \rangle}} = \frac{1}{\sqrt{\langle [h_1(\mathbf{r}_1', \tau_1) - h_1(\mathbf{r}_2', \tau_2)]^2 \rangle}}. \quad (69)$$

Next we substitute the generic dynamical scaling form in Eq. (67) for the correlation function in the denominator of Eq. (69) and use the resulting expression of $\overline{\langle v \cdot v \rangle}$ on the rhs of Eq. (48). The subsequent evaluation of the integral $I_2(t_1, t_2)$ for large t_1 and t_2 followed by the evaluation of the incremental correlation function $C_2(t_1, t_2, t_0)$ can be done by following an identical analysis as in Sec. V B which we do not repeat here. After performing these steps one finds that for large $t_0 \to \infty$

$$C_2(t_1, t_2, t_0 \to \infty) \sim b_1 |t_2 - t_2|^{1 - d/2} + c_1 |t_2 - t_1|^{2 - \beta_1}, \quad (70)$$

where b_1 and c_1 are unimportant constants. Thus h_2 satisfies the defining property in Eq. (30) of a FBM with a Hurst exponent given by

$$H_2 = \max[(2 - d)/4, 1 - \beta_1/2]. \quad (71)$$

In particular, for $\beta_1 < 1 + d/2$ (which seems to be the case for most interfaces), one gets $H_2 = 1 - \beta_1/2$. This then leads to the persistence exponent

$$\theta_s^2 = 1 - H_2 = \beta_1/2. \quad (72)$$

For example, for a $(1+1)$-dimensional KPZ interface h_1 for which $\beta_1 = 1/3$, one gets

$$H_2 = 1 - \beta_1/2 = 5/6; \quad \theta_s^2 = 1/6. \quad (73)$$

VII. NUMERICAL SIMULATIONS

In this section we numerically verify some of the analytical predictions made in the previous sections for the coupled interface model, in particular the fact that in the $t_0 \to \infty$ limit h_2 is generically a FBM with Hurst exponent $H_2 = 1 - \beta_1/2$, i.e., for large $|t_2 - t_1|$,

$$C_2(t_1, t_2, t_0 \to \infty) = \lim_{t_0 \to \infty} E[(h_2(\mathbf{r}, t_0 + t_1) - h_2(\mathbf{r}, t_0 + t_2))^2]$$

$$\sim |t_2 - t_1|^{2 - \beta_1}, \quad (74)$$

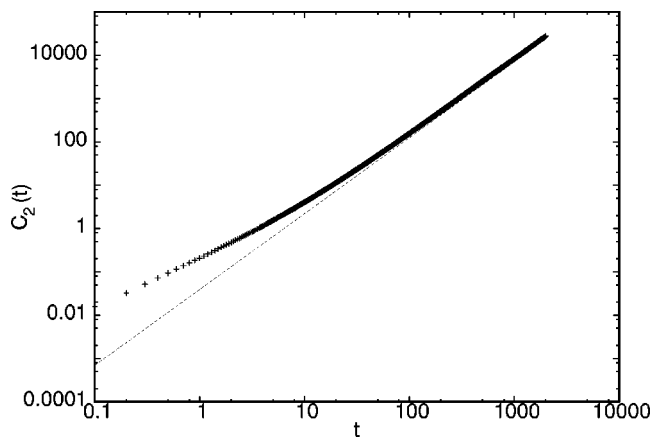where β_1 is the growth exponent of the first surface h_1. We

FIG. 2. Log-log plot of the incremental correlation function $C_2(|t_2-t_1|=t)$ vs t for the Rouse model. The chain length is $N=1024$, while $\Delta t=0.1$ and $\Delta x=0.5$; number of disorder histories $=30$ and thermal histories$=30$. The numerical data (shown by $+$ signs) is compared to the theoretical prediction of a power law with power $7/4$ as shown by the straight line.

have checked this prediction numerically in $d=1$ for two cases: (i) when the first surface evolves via the Edwards-Wilkinson equation so that $\beta_1=1/4$ and Eq. (74) predicts $C_2(t_1,t_2,t_0\to\infty)\sim|t_2-t_1|^{7/4}$ for large $|t_2-t_1|$. This case corresponds also to the Rouse chain advected by a transverse velocity field as mentioned in Sec. III B and (ii) when the first surface evolves via the KPZ equation so that $\beta_1=1/3$. In this case, Eq. (74) predicts $C_2(t_1,t_2,t_0\to\infty)\sim|t_2-t_1|^{5/3}$. The results for the simulations in the two cases are shown in Figs. 2 and 3, respectively.

Our simulation techniques are straightforward. For case (i) above we use the time discretized version of the Rouse chain model, i.e., Eqs. (20) and (21). We set $t_m=m\Delta t$ and rewrite Eqs. (20) and (21) as
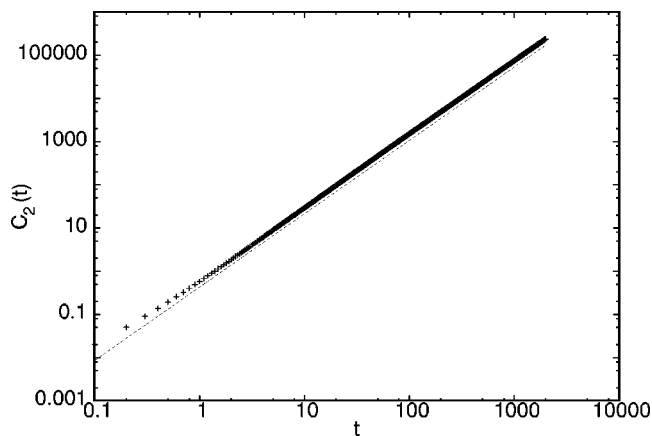

FIG. 3. Log-log plot of the incremental correlation function $C_2(|t_2-t_1|=t)$ vs t when h_1 evolves via the $(1+1)$-dimensional KPZ equation. The chain length is $N=4096$, while $\Delta t=0.1$ and $\Delta x=0.5$. The theoretical prediction is the straight line that corresponds to a power law in t with a power $5/3\simeq1.67$, while a straight line fit to the numerical data (shown by $+$ signs) gives an exponent ≈1.695. This slight discrepancy is commented upon in the text.

$$x_n(t_{m+1}) = x_n(t_m) + \Delta t[x_{n+1}(t_m) + x_{n-1}(t_m) - 2x_n(t_m)]$$
$$+ \sqrt{\Delta t}\,\eta_1(n,t_m), \qquad (75)$$

$$y_n(t_{m+1}) = y_n(t_m) + \Delta t[y_{n+1}(t_m) + y_{n-1}(t_m) - 2y_n(t_m)]$$
$$+ \Delta t v(x_n(t_m)) + \sqrt{\Delta t}\,\eta_2(n,t_m). \qquad (76)$$

For the boundary points $n=1$ and $n=N$, we use free boundary conditions, i.e., we hold $x_0=x_1$, $y_0=y_1$, $x_N=x_{N+1}$, and $y_N=y_{N+1}$ for all times t_m. We choose $\Delta t<0.5$ in our simulations so that the stability is guaranteed [20]. The variables $\eta_1(n,t_m)$ and $\eta_2(n,m)$ are independent Gaussian variables for all n and t_m and each is distributed with zero mean and unit variance. We have checked that even if the noise variables have a binary distribution (i.e., $+1$ and -1 each with probability $1/2$), the results at long times do not change. Besides, as it turns out from Eq. (70) that the thermal noise η_2 is actually irrelevant for the long time properties of the h_2 process, we have dropped η_2 in Eq. (76) in most of our simulations.

We choose the random quenched transverse velocity $v(x)$ in the following way. First we consider a grid along the x direction with grid spacing Δx. In fact this grid represents the layered structure of the velocity field. At each point of this grid we choose independently $v(x)=u(x)/\sqrt{\Delta x}$ where $u(x)$ is a Guassian random variable with zero mean and unit variance. Once a set of $\{v(x)\}$ is thus chosen, they remain fixed at all times during different thermal histories. This set $\{v(x)\}$ constitutes a particular realization of disorder. Finally one performs the disorder average $\overline{(\cdots)}$ by averaging over various realizations of the set $\{v(x)\}$. Now, x_n in Eq. (75) at any given time is usually a continuous variable and may not correspond to a grid point. In fact, in general, x_n will be between two grid points, say x_0 and $x_0+\Delta x$. In such a case, we use, as a convention, $v(x_0)$ while simulating the rhs of Eq. (76). For a fixed realization of the disorder $\{v(x)\}$, we average over 30–40 thermal histories (generated via η_1) and then a final average is done over 30–40 disorder realizations.

Using this discretization scheme we have computed the incremental correlation function $C_2(t_1,t_2,t_0)$ as defined in Eq. (74) beyond some large waiting time t_0 (typically $t_0 \sim 30\,000$ steps) for different choices of time step Δt and the grid size Δx. We have checked that the results do not vary much with the time step or the grid size. We find that, as predicted analytically, $C_2(t_1,t_2,t_0\to\infty)$ depends only on the time difference $|t_2-t_1|$ and for large $|t_2-t_1|$, C_2 scales as a power law with an exponent ≈1.75 which is consistent with the theoretical prediction $7/4$. A plot of C_2 as a function of $|t_2-t_1|$ is shown in Fig. 2. This confirms numerically the theoretical prediction that indeed h_2 is a FBM with the Hurst exponent $H_2=7/8$ and hence also confirms that the persistence exponent $\theta_s^2=1/8$.

We also numerically computed the incremental correlation function $C_2(t_1,t_2,t_0\to\infty)$ for case (ii), i.e., when h_1 evolves by the $(1+1)$-dimensional KPZ equation. For the $(1+1)$-dimensional KPZ equation in Eq. (66), one needs to be careful about the discretization of space and time as has been discussed extensively in the literature [43–47]. In this

paper we used the discretization scheme proposed by Newman and Swift [45] with periodic boundary condition,

$$h_1(x_n,t_{m+1}) = \max[h_1(x_{n+1},t_m),h_1(x_{n-1},t_m),h_1(x_n,t_m)]$$
$$+ \sqrt{\Delta t}\,\eta_1(x_n,t_m). \quad (77)$$

We show in Fig. 3 the incremental correlation function $C_2(|t_2-t_1|=t,t_0\to\infty)$ vs t. Evidently $C_2(t)$ increases as a power law with an exponent ≈ 1.695 which is close to the theoretical prediction $5/3 \simeq 1.67$. The slight discrepancy is due to the system size used by us, for which even the scaling exponent β_1 for h_1 actually is not exactly $1/3$ but close to ≈ 0.305. We have checked that the exponent systematically approaches the expected value with increasing N, and here our quoted value is based on the largest N that we could simulate.

VIII. CONCLUSION

In this paper we have studied the time-dependent properties in a simple model of coupled interfaces characterized by heights h_1 and h_2, respectively, growing over a d-dimensional substrate. The evolution of the first interface h_1 is not affected by h_2. In fact, h_1 can be any generic growing interface characterized by a growth exponent β_1. For example, h_1 may be evolving either via the Edwards-Wilkinson equation or via the Kardar-Parisi-Zhang equation. The evolution of the second interface h_2, however, is coupled to that of h_1 by a transverse quenched random velocity field, in addition to having the usual Edwards-Wilkinson dynamics. In the limiting case $d\to 0$, our model reduces to the Matheron–de Marsily model where one studies the motion of a Brownian particle in a two-dimensional plane in the presence of a transverse velocity field. In the limit $d=1$, our model describes the equations of motion of a Rouse polymer chain in two dimensions in presence of a transverse velocity field.

We have obtained analytical results for the persistence properties in this model. The main result of this paper is to show analytically that after a long waiting time ($t_0\to\infty$), the process h_2, at a fixed point in space but as a function of time, becomes a fractional Brownian motion with a Hurst exponent $H_2 = 1 - \beta_1/2$. By using a known first-passage property of fractional Brownian motion we have then shown that after a long waiting time $t_0\to\infty$, the persistence probability $P_2(t)$ that the process h_2 at a fixed point in space does not come back to its value at t_0 over the time interval $[t_0,t_0+t]$ decays as a power law for large t, $P_2(t)\sim t^{-\theta_s^2}$ where the exponent $\theta_s^2 = 1 - H_2 = \beta_1/2$. For $d=1$, these analytical predictions have been verified numerically in two cases: when h_1 evolves via the Edwards-Wilkinson equation and when h_1 evolves via the Kardar-Parisi-Zhang equation.

The mapping of a relevant stochastic process in some limits to a fractional Brownian motion is a rather powerful technique for studying the first-passage properties for non-Gaussian and/or non-Markovian processes. The mapping does not work always, but if it works one can use the known first-passage property of the fractional Brownian motion. This technique has been used successfully in a number of contexts previously [13,20–22,26,27]. We have demonstrated that the same technique also works in another class of coupled interface models discussed in this paper. It would be interesting to find other such cases where one can apply the same technique successfully. Finally, it would be interesting to study the persistence properties in more realistic models of coupled interfaces that are closer to the experimental situation of fluctuating steps [28].

[1] I. F. Blake and W. C. Lindsay, IEEE Trans. Inf. Theory **19**, 295 (1973).

[2] S. Redner, *A Guide to First-passage Processes* (Cambridge University Press, Cambridge, England, 2001).

[3] B. Derrida, A. J. Bray, and C. Godrèche, J. Phys. A **27**, L357 (1994).

[4] For a brief review on persistence, see S. N. Majumdar, Curr. Sci. **77**, 270 (1999).

[5] S. N. Majumdar, C. Sire, A. J. Bray, and S. J. Cornell, Phys. Rev. Lett. **77**, 2867 (1996); B. Derrida, V. Hakim, and R. Zeitak, *ibid.* **77**, 2871 (1996).

[6] C. M. Newman and D. L. Stein, Phys. Rev. Lett. **82**, 3944 (1999).

[7] S. Jain, Phys. Rev. E **60**, R2445 (1999).

[8] A. Comtet and D. S. Dean, J. Phys. A **31**, 8595 (1998).

[9] F. Igloi and H. Reiger, Phys. Rev. E **58**, 4238 (1998); Europhys. Lett. **45**, 673 (1999).

[10] D. S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. Lett. **80**, 3539 (1998); Phys. Rev. E **59**, 4795 (1999).

[11] S. N. Majumdar and A. Comtet, Phys. Rev. E **66**, 061105 (2002).

[12] S. Redner, Phys. Rev. E **56**, 4967 (1997).

[13] S. N. Majumdar, Phys. Rev. E **68**, 050101 (2003).

[14] A. J. Bray and P. Gonos, J. Phys. A **37**, L361 (2004).

[15] G. Matheron and G. de Marsily, Water Resour. Res. **16**, 901 (1980).

[16] P. E. Rouse, J. Chem. Phys. **21**, 1272 (1953).

[17] G. Oshanin and A. Blumen, Phys. Rev. E **49**, 4185 (1994); Macromol. Theory Simul. **4**, 87 (1995).

[18] K. J. Wiese and P. Le Doussal, Nucl. Phys. B **552**, 529 (1999).

[19] S. Jespersen, G. Oshanin, and A. Blumen, Phys. Rev. E **63**, 011801 (2001).

[20] J. Krug, H. Kallabis, S. N. Majumdar, S. J. Cornell, A. J. Bray, and C. Sire, Phys. Rev. E **56**, 2702 (1997).

[21] H. Kallabis and J. Krug, Europhys. Lett. **45**, 20 (1999).

[22] S. N. Majumdar and A. J. Bray, Phys. Rev. Lett. **86**, 3700 (2001).

[23] M. Constantin *et al.*, Phys. Rev. Lett. **91**, 086103 (2003).

[24] C. Dasgupta, M. Constantin, S. Das Sarma, and S. N. Majumdar, Phys. Rev. E **69**, 022101 (2004).

[25] M. Constantin, S. Das Sarma, and C. Dasgupta, Phys. Rev. E **69**, 051603 (2004); M. Constantin and S. Das Sarma, *ibid.* **69**,

052601 (2004).

[26] M. Constantin, C. Dasgupta, P. Punyindu Chatraphorn, S. N. Majumdar, and S. Das Sarma, Phys. Rev. E **69**, 061608 (2004).

[27] J. Krug, Physica A **340**, 647 (2004).

[28] D. B. Dougherty *et al.*, Phys. Rev. Lett. **89**, 136102 (2002); O. Bondurchuk *et al.*, cond-mat/0408181.

[29] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).

[30] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[31] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).

[32] J.-P. Bouchaud, A. Georges, J. Koplik, A. Provata, and S. Redner, Phys. Rev. Lett. **64**, 2503 (1990).

[33] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).

[34] S. Redner, Physica A **168**, 551 (1990).

[35] H. P. McKean, J. Math. Kyoto Univ. **2**, 227 (1963); M. Goldman, Ann. Math. Stat. **42**, 2150 (1971); Y. G. Sinai, Theor. Math. Phys. **90**, 219 (1992); T. W. Burkhardt, J. Phys. A **26**, L1157 (1993).

[36] S. Redner and P. L. Krapivsky, J. Stat. Phys. **82**, 999 (1996).

[37] B. B. Mandelbrot and J. W. van Ness, SIAM Rev. **10**, 422 (1968).

[38] S. M. Berman, Ann. Math. Stat. **41**, 1260 (1970).

[39] A. Hansen, T. Engoy, and K. J. Maloy, Fractals **2**, 527 (1994).

[40] M. Ding and W. Yang, Phys. Rev. E **52**, 207 (1995).

[41] S. N. Majumdar and A. J. Bray, Phys. Rev. Lett. **91**, 030602 (2003).

[42] G. F. Newell and M. Rosenblatt, Ann. Math. Stat. **33**, 1306 (1962).

[43] C. Dasgupta, S. Das Sarma, and J. M. Kim, Phys. Rev. E **54**, R4552 (1996).

[44] T. J. Newman and A. J. Bray, J. Phys. A **29**, 7917 (1996).

[45] T. J. Newman and M. R. Swift, Phys. Rev. Lett. **79**, 2261 (1997).

[46] C.-H. Lam and F. G. Shin, Phys. Rev. E **57**, 6506 (1998); **58**, 5592 (1998).

[47] S. N. Majumdar and A. Comtet, e-print cond-mat/0409566.